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integral can be explicitly evaluated in terms of the commoner functions) then as the mean f_Ω of $f(\psi)$ for this distribution of ψ over $\Omega[\psi]$ we have

$$f_\Omega=\frac{I_1}{I_2}. \tag{3}$$

As has appeared time after time in all periodicals having a department devoted to mean values, when a problem is proposed in the indefinite form (A), different solvers introduce different laws of distribution, [each one the law appearing to him the most natural, or often enough, the law for which he can explicitly evaluate the integrals (2)]. These laws of distribution should be introduced explicitly as needed to make the problem definite. The possible laws of distribution are without number.

There is *no such thing* as THE *correct* solution of a problem stated in the indefinite form (A).

I make these quite obvious remarks for the sake of those persons who enjoy plunging into the whirl of the integrations without due meditation on the essential nature of the problem they are attacking.

The University of Chicago, November 5, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from September-October Number.]

SYSTEMS OF INTRANSITIVITY 3, 3.

(a) By multiplying every substitution of one group by every substitution of the other group we obtain

1. A group of order 36, viz:

1	1*
abc	*def*
acb	*dfe*
ab	*de*
ac	*df*
bc	*ef*

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*This notation indicates that every substitution on one side of the line is multiplied into every one on the other side.

2. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \\ & de \\ & df \\ & ef \end{array}$$

3. A group of order 9, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \end{array}$$

(*b*) By making the identical substitutions in the two systems correspond we obtain

4. A group of order 6, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \\ ab.de \\ ac.df \\ bc.ef \end{array}$$

5. A group of order 3, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \end{array}$$

(*c*) By multiplying the divisions according to a self-conjugate subgroup of one group into the corresponding divisions of the other we obtain

6. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1\ * \\ abc & def \\ acb & dfe \\ \hline ab & de \\ ac & df \\ bc & ef \end{array}$$

***This notation implies that the divisions of one column are to be multiplied into the corresponding divisions of the other column. Thus, *ab* is multiplied into *de*, *df*, and *ef*.**

SYSTEMS OF INTRANSITIVITY 2, 4.

There are five transitive groups of degree four, viz :

$$(abcd)all,\ (abcd)pos,\ (abcd)_8,\ (abcd)_4,\ (abcd)cyc$$

$$(abcd)_8 = \begin{array}{llll} 1 & ac & ab.cd & abcd \\ & bd & ac.bd & adcb \\ & & ad.bc & \end{array}$$

$$(abcd)_4 = \begin{array}{ll} 1 & ab.cd \\ & ac.bd \\ & ad.bc \end{array} \qquad (abcd)cyc = \begin{array}{lll} 1 & ac.bd & abcd \\ & & adcb \end{array}$$

By multiplying every substitution in each of these groups by $1, ef$ we clearly obtain five additional intransitive groups of degree six whose orders are 48, 24, 16, 8, and 8 respectively. It remains to find the groups which can be obtained by multiplying the self conjugate subgroups which include half the substitutions of these groups by 1 and the remaining substitutions by *ef*.

Since $(abcd)all$ and $(abcd)cyc$ have one such self conjugate subgroup apiece, they give rise to two groups of the required type. $(abcd)_8$ and $(abcd)_4$ have three such subgroups apiecc, but in the latter case they are all conjugate, as we have already seen ; hence we obtain the following four groups from these two groups :

$$\begin{array}{c} (1) \\ \begin{array}{r|l} 1 & \\ ac & \\ bd & 1 \\ ac.bd & \\ \hline ab.cd & \\ ad.bc & ef \\ abcd & \\ adcb & \end{array} \end{array} \qquad \begin{array}{c} (2) \\ \begin{array}{r|l} 1 & \\ ab.cd & \\ ac.bd & 1 \\ ad.bc & \\ \hline ac & \\ bd & ef \\ abcd & \\ adcb & \end{array} \end{array} \qquad \begin{array}{c} (3) \\ \begin{array}{r|l} 1 & \\ ac.bd & \\ abcd & 1 \\ adcb & \\ \hline ac & \\ bd & ef \\ ab.cd & \\ ad.bc & \end{array} \end{array} \qquad \begin{array}{c} (4) \\ \begin{array}{r|l} 1 & \\ ab.cd & 1 \\ \hline ac.bd & ef \\ ad.bc & \end{array} \end{array}$$

We have thus found that there are eleven intransitive groups of degree six whose systems of intransitivity are 2 and 4.

SYSTEMS OF INTRANSITIVITY 2, 2, 2.

If we restrict ourselves to the first two systems we obtain an intransitive group of degree four. Our problem is thus reduced to the finding all the intransitive groups that can be obtained by combining one of the two intransitive groups of degree four with the group $1, ef$.

By multiplying every substitution of these two groups into $1, ef$ we obtain two groups whose degrees are four and eight respectively. By multiplying one into the self conjugate subgroups of these groups and ef into the remaining substitutions we obtain the following additional groups :

$$\begin{array}{r|l} 1 & 1 \\ \hline ac.bd & ef \end{array} \qquad \text{and} \qquad \begin{array}{r|l} 1 & \\ ac.bd & 1 \\ \hline ac & ef \\ bd & \end{array}$$

We have now worked over the entire field and found twenty-one intransitive groups of degree six. These may be written as follows :*

LIST OF INTRANSITIVE GROUPS OF DEGREE SIX.

Order	No.	Group
2	1	$(ac.bd.ef)$
3	1	$(abc.def)$cyc
4	1	$(ab.cd)(ef)$
	2	$\{(ab)(cd)(ef)\}$pos
	3	$\{(abcd)$cyc$(ef)\}$pos
	4	$\{(abcd)_4(ef)\}$dim
6	1	$(abc.def)$all
8	1	$(ab)(cd)(ef)$
	2	$(abcd)$cyc(ef)
	3	$(abcd)_4(ef)$
	4—6	$\{(abcd)_8(ef)\}$
9	1	(abc)cyc(def)cyc
16	1	$(abcd)_8(ef)$
18	1	(abc)all(def)cyc
	2	$\{(abc)$all(def)all$\}$pos
24	1	$\{(abcd)$all$(ef)\}$pos
	2	$(abcd)$pos(ef)
36	1	(abc)all(def)all
48	1	$(abcd)$all(ef)

From what has been said it may be seen that intransitive groups of a given degree n can be readily constructed provided the transitive groups degree k are known, where $k=2, 3, 4, \ldots n-2$. When n does not exceed 9 it is not difficult to write down all the possible intransitive groups, but for larger values of n the number of groups is so large that the construction of all such groups becomes comparatively quite laborious. The transitive groups seem much more important than the intransitive ones and we proceed to give methods by which they may be constructed.

Definition.—If a transitive group is such that any β letters may be replaced

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*Cf. Professor Cayley : *Quarterly Journal of Mathematics*, vol. 25, pp. 71—79.

by any required set of β letters taken in any one of the $\beta!$ possible ways it is said to be β-fold transitive. If $\beta=1$ the group is one-fold or *simply* transitive.

Definition.—A transitive group whose letters can be divided into systems such that all the substitutions of the group perform upon these systems only one or both of the operations (1) interchanging the systems as units and (2) permuting the letters in the systems, is called a *non-primitive group.** All other transitive groups are called *primitive.*

We shall consider separately the construction of these two classes of transitive groups.

CONSTRUCTION OF THE NON-PRIMITIVE GROUPS.

Since the non-primitive group must be transitive it is always possible to replace any system by any required system by means of some substitution of the group. Hence it follows that the same number of letters is found in each system.

If $G=s_1, s_2, \ldots., s_g$ is any non-primitive group and $G_1=s_1, s_2, \ldots., s_{g_1}$ are the substitutions of G which do not interchange any of the systems then will G_1 be an *intransitive self conjugate subgroup of G.*

G_1 is a group because the substitutions

$$s_\alpha s_\beta \quad \begin{matrix} \alpha=1, 2, \ldots. g_1 \\ \beta=1, 2, \ldots. g_1 \end{matrix}$$

cannot interchange any of the systems. They must therefore be found in G_1 since they certainly belong to G.

G_1 is a self conjugate subgroup of G because

$$s_\gamma^{-1} G_1 s_\gamma \quad g_1<\gamma<g+1$$

must be a subgroup of G, which does not interchange the systems (the interchange effected by s_γ^{-1} being restored by s_γ) and must therefore be G_1 itself.

We may suppose that the transitive constituents of G_1 form the systems. For if $a_1, a_2, \ldots., a_e$ and $b_1, b_2, \ldots., b_k$ are transitively connected by G_1 then G contains some substitution s_γ which replaces a_1 by b_1. We proved above that

$$s_\gamma^{-1} s_\alpha s_\gamma = s_\beta \quad \begin{matrix} \alpha=1,2, \ldots., g_1 \\ \beta=1,2, \ldots., g_1 \end{matrix}$$

Suppose s_α so chosen that it replaces a_1 by a_2. Then it follows that s_γ replaces a_2 by one of the given b's. Similarly, we may show that s_γ replaces all of the given a's by the given b's. Since the a's and b's can be interchanged in this course of reasoning we have that $e=k$, or, more generally speaking, that

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*Such a group must be *simply* transitive, for if $\beta>1$ it would be possible to replace two letters of the same system by letters of different systems.

$$s_\gamma \qquad g<\gamma<g+1$$

either does not interchange any elements of two transitive constituents of G_1 or it interchanges all. Hence the transitive constituents satify the definition of systems.*

We are now prepared to see that the construction of non-primitive groups consists of two operations :

1. The construction of an intransitive group G_1 whose transitive constituents are conjugate groups.†
2. The construction of the substitutions of G which interchange the systems.

The first of these operations is a special case under the construction of intransitive groups and needs therefore no further attention. With respect to the second we shall first consider some special cases and then take up the general problem.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from September-October Number.]

SCHOLION II, *in which is weighed the idea of that brilliant man Giovanni Alfonso Borelli in his Euclides Restitutus.*

This most learned author blames Euclid, because he defines parallel straight lines to be those, *which lying in the same plane do not meet on either side, even if produced into the infinite.*

He offers as ground for his accusation, that such relation is unknown: *first*, he says, *because we are ignorant whether such infinite non-concurrent lines can be found in nature; then also because we cannot perceive the properties of the infinite, and hence a relation of this sort is not clearly cognized.*

But with reverence for so great a man it may be said: can Euclid be blamed, because (to bring forward one among innumerable examples) he de-

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*Cf. Netto's Theory of Substitutions (Cole's edition), §67.

†It has not been proved that all groups of this form can be used for G_1 but that every G_1 is of this form. The former cannot be proved.